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# INTERIOR ESTIMATES IN CAMPANATO SPACES RELATED TO QUADRATIC FUNCTIONALS

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## Abstract

In this paper we obtain interior estimates in Campanato spaces for the derivatives of the minimizers of quadratic functionals.

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## 1 Introduction and Preliminary Tools

The papers concerned with the regularity problem almost always have as a common starting point the Euler's equation related to a generic functional  $I$ . In the paper by Giaquinta and Giusti [11], the authors investigate the hölder continuity of the minima working directly with the functional  $I$  instead of Euler's equation.

In the present paper, following the method in [11], we have studied regularity properties of the minima of variational integrals of the type:

$$\mathcal{A}(u, \Omega) = \int f(x, u, Du) dx,$$

$\Omega \subset \mathbb{R}^n, n \geq 3$  is a bounded opens set  $u : \Omega \rightarrow \mathbb{R}^N, u(x) = (u^1(x), \dots, u^N(x)),$   
 $Du = D_\alpha u^i, D_\alpha = \frac{\partial}{\partial x^\alpha}, i = 1, \dots, N, \alpha = 1, \dots, n,$

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \rightarrow \mathbb{R}$$

defined by

$$f(x, u, Du) = A_{ij}^{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^j + g(x, u, Du).$$

Let us now give the following definitions useful in the sequel.

**Definition 1.1.** (see [16], [20]). Let  $1 \leq p < \infty, 0 \leq \lambda < n$ .

By  $L^{p,\lambda}(\Omega)$  we denote the linear space of functions  $f \in L^p(\Omega)$  such that

$$\|f\|_{L^{p,\lambda}(\Omega)} = \left\{ \sup_{\substack{0 < \rho < \text{diam } \Omega \\ x \in \Omega}} \rho^{-\lambda} \int_{\Omega \cap B(x,\rho)} |f(y)|^p dy \right\} < +\infty$$

where  $B(x, \rho)$  ranges in the class of the balls of  $\mathbb{R}^n$  of radius  $\rho$  around  $x$ .

We have that  $\|f\|_{L^{p,\lambda}(\Omega)}$  is a norm respect to which  $L^{p,\lambda}(\Omega)$  is a Banach space, and also that

$$\|f\|_{L^p(\Omega)} \leq (\text{diam } \Omega)^{\frac{\lambda}{p}} \|f\|_{L^{p,\lambda}(\Omega)}.$$

Before the definition of the Campanato spaces let us define  $f_{B(x,\rho)}$  as the integral average

$$f_{B(x,\rho)} = \frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} f(y) dy$$

of the function  $f(x)$  over the balls  $B(x, \rho)$  of  $\mathbb{R}^n$ . When no confusion may arise, we will write  $f_{B_\rho}$  or  $f_\rho$  instead of  $f_{B(x,\rho)}$ .

**Definition 1.2.** (see e. g. [3], [9]). Let  $1 \leq p < \infty$  and  $\lambda \geq 0$ .

By Campanato spaces  $\mathcal{L}^{p,\lambda}(\Omega)$  we denote the linear space of functions  $u \in L^p(\Omega)$  such that

$$[f]_{p,\lambda} = \left\{ \sup_{x \in \Omega, 0 < \rho < \text{diam } \Omega} \rho^{-\lambda} \int_{\Omega \cap B(x,\rho)} |f(y) - f_\rho|^p dy \right\}^{\frac{1}{p}} < +\infty.$$

$\mathcal{L}^{p,\lambda}(\Omega)$  are Banach spaces with the following norm

$$\|f\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{p,\lambda}$$

which simply demonstrate that  $u \in \mathcal{L}^{p,\lambda}(\Omega)$  if and only if

$$\sup_{x \in \Omega, 0 < \rho < \text{diam } \Omega} \rho^{-\lambda} \inf_{c \in \mathbb{R}} \int_{\Omega \cap B(x, \rho)} |f - c|^p dy < \infty.$$

Using Hölder inequality we have that

$$\mathcal{L}^{p_1, \lambda_1}(\Omega) \subset \mathcal{L}^{p, \lambda}(\Omega)$$

where

$$p \leq p_1, \quad \frac{n - \lambda}{p} \geq \frac{n - \lambda_1}{p_1}.$$

Let us observe that

$$\int_{\Omega \cap B(x, \rho)} |f - f_\rho|^p dy \leq C \cdot \int_{\Omega \cap B(x, \rho)} \left( |f|^p + |\Omega \cap B(x, \rho)| \cdot |f_\rho|^p \right) dy$$

and also

$$|f_\rho|^p \leq \frac{1}{|\Omega \cap B(x, \rho)|} \cdot \int_{\Omega} |f|^p dy.$$

Below, we consider  $0 \leq \lambda < n$ .

We use

$$[f]_{p, \lambda} \leq C \|f\|_{L^{p, \lambda}(\Omega)}.$$

to obtain the following relation between Morrey and Campanato spaces

$$L^{p, \lambda}(\Omega) \subset \mathcal{L}^{p, \lambda}(\Omega). \quad (1.1)$$

Let us now recall the definitions of the *BMO* and *VMO* classes.

**Definition 1.3.** (see [15]). We say that a function  $f$  belongs to the John-Nirenberg space *BMO*, or that  $f$  has "bounded mean oscillation", if

$$\|f\|_* \equiv \sup_{B_\rho \subset \mathbb{R}^n} \frac{1}{|B_\rho|} \int_{B_\rho} |f(y) - f_\rho| dy < \infty$$

where  $f_\rho$  is the integral average of the function  $f$  over the balls  $B_\rho$ .

Let us define, for a function  $f \in BMO$ ,

$$\eta(r) = \sup_{x \in \mathbb{R}^n, \rho \leq r} \frac{1}{|B_\rho|} \int_{B_\rho} |f(x) - f_\rho| dx.$$

**Definition 1.4.** (see [22]). A function  $f \in BMO$  belongs to the class  $VMO$ , or  $f$  has "vanishing mean oscillation" if

$$\lim_{r \rightarrow 0^+} \eta(r) = 0.$$

We are now ready to formulate the hypothesis on the terms  $A_{ij}^{\alpha\beta}(x, u)$  and  $g(\cdot, u, Du)$ .

We suppose that  $A_{ij}^{\alpha\beta}(x, u)$  are bounded functions in  $\Omega \times \mathbb{R}^N$ , such that:

$$(A1) \quad A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}.$$

$$(A2) \quad \text{For every } u \in \mathbb{R}^N, A_{ij}^{\alpha\beta}(\cdot, u) \in VMO(\Omega).$$

$$(A3) \quad \text{For every } x \in \Omega \text{ and } u, v \in \mathbb{R}^N,$$

$$|A_{ij}^{\alpha\beta}(x, u) - A_{ij}^{\alpha\beta}(x, v)| \leq \omega(|u - v|^2)$$

for some monotone increasing concave function  $\omega$  with  $\omega(0) = 0$ .

$$(A4) \quad \text{There exists a positive constant } \nu \text{ such that}$$

$$\nu |\xi|^2 \leq A_{ij}^{\alpha\beta}(x, u) \xi_\alpha^i \xi_\beta^j$$

for a.e.  $x \in \Omega$ , all  $u \in \mathbb{R}^N$  and  $\xi \in \mathbb{R}^{nN}$ .

We suppose that the function  $g$  is a Charathéodory function, that is:

$$(g1) \quad g(\cdot, u, Du) \text{ is measurable in } x \quad \forall u \in \mathbb{R}^N, \forall z \in \mathbb{R}^{nN};$$

$$(g2) \quad g(x, \cdot, \cdot) \text{ is continuous in } (u, z) \text{ a. e. } x \in \Omega;$$

moreover we consider  $g$  satisfying the condition:

$$(g3)$$

$$|g(x, u, z)| \leq g_1(x) + H |z|^\gamma,$$

$$g_1 \geq 0, \text{ a. e. in } \Omega, g_1 \in L^p(\Omega), 2 < p \leq \infty, H \geq 0, 0 \leq \gamma < 2.$$

We point out that since  $C^0$  is a proper subset of  $VMO$ , the continuity of  $A_{ij}^{\alpha\beta}(x, u)$  with respect to  $x$  is not assumed.

Let us make some remarks on  $VMO$  class. It was at first defined by Sarason in 1975 and later it was considered by many others. For instance we recall the papers by Chiarenza, Frasca and Longo [4] where the authors

answer a question raised thirty years before by C. Miranda in [18]. In his paper he considers a linear elliptic equation where the coefficients  $a_{ij}$  of the higher order derivatives are in the class  $W^{1,n}(\Omega)$  and asks whether the gradient of the solution is bounded, if  $p > n$ . Chiarenza, Frasca and Longo suppose  $a_{ij} \in VMO$  and prove that  $Du$  is hölder continuous for all  $p \in ]1, +\infty[$ . We point out that  $W^{1,n} \subset VMO$  because, using Poincaré's inequality

$$\int_B |f(x) - f_B| \leq c(n) \left( \int_B |\nabla u| dx \right)^{\frac{1}{n}}$$

and the term on the right-hand side tending to zero as  $|B| \rightarrow 0$ . Later the interior estimates obtained by Chiarenza, Frasca and Longo were extended to boundary estimates in [5]. From these papers on, many authors have used this space  $VMO$  to obtain regularity results for partial differential equations and systems with discontinuous coefficients. We recall for example Bramanti and Cerutti [2] for parabolic equations and many others.

With this useful assumption we investigate the regularity of the minimizers for the quadratic functional. Its existence is guaranteed, being the functional  $\mathcal{A}$  sequentially lower semicontinuous with respect to the  $H^{1,2}$ -weak topology (see [10]).

## 2 Main Results

**Theorem 2.5.** *Let  $u \in W^{1,2}(\Omega, \mathbb{R}^N)$  be a minimum of the functional  $\mathcal{A}(u, \Omega)$  defined above. Suppose that assumptions (A-1), (A-2), (A-3), (A-4),  $1 < q \leq 2$  and (g-1), (g-2) and (g-3) are satisfied. Then for  $\lambda = n(1 - \frac{2}{p})$  we have*

$$Du \in \mathcal{L}_{loc}^{q,\lambda}(\Omega_0, \mathbb{R}^{nN}) \quad (2.2)$$

where

$$\Omega_0 = \left\{ x \in \Omega : \liminf_{R \rightarrow 0} \frac{1}{R^{n-2}} \int_{B(x,R)} |Du(y)|^2 dy = 0 \right\}.$$

The set  $\Omega_0$  is obligatory, in fact when we pass from the regularity theory for scalar minimizers of solutions of elliptic equations to the regularity theory for vector-valued minimizers of solutions of elliptic systems, the situation changes completely: regularity is an exceptional occurrence everywhere, excluding the two dimensional case. In 1968 De Giorgi in [7] showed that his

regularity result for solutions of second order elliptic equations with measurable bounded coefficients cannot be extended to solutions of elliptic systems. He presented the quadratic functional

$$\mathcal{S} = \int_{\Omega} A_{ij}^{\alpha\beta}(x) D_{\alpha} u^i D_{\beta} u^j dx$$

with  $A_{ij}^{\alpha\beta} \in L^{\infty}(\Omega)$ , such that

$$\exists \nu > 0 : A_{ij}^{\alpha\beta} \chi_{\alpha}^i \chi_{\beta}^j \geq \nu |\chi|^2 \text{ p.e. } x \in \Omega, \forall \chi \in \mathbb{R}^{nN}$$

De Giorgi proves that  $\mathcal{S}$  has a minimizer that is a function having a point of discontinuity in the origin. Later, Souček in [23] showed that minimizers of functionals of the type  $\mathcal{S}$  can be discontinuous, not only in a point, but also on a dense subset of  $\Omega$ . Modifying De Giorgi's example, Giusti and Miranda in [13] showed that solutions of elliptic quasilinear systems of the type

$$\int_{\Omega} A_{ij}^{\alpha\beta}(u) D_{\alpha} u^i D_{\beta} \varphi^j dx = 0, \forall \varphi \in C_0^{\infty}(\Omega, \mathbb{R}^N)$$

with analytic elliptic coefficients  $A_{ij}^{\alpha\beta}$  have singularities in dimension  $n \geq 3$ . We observe that we can get global regularity for some special cases, see for example [24].

Similar examples were presented in the meantime independently by Maz'ya in [17]. Even Giaquinta in 1993 in [10], Morrey in [19] and others were interested in these problems of solutions of elliptic systems, solutions in general non regular. Then we can prove regularity except on a set, hopefully not too large.

For linear systems, regularity results assuming  $A_{ij}^{\alpha\beta}$  constant or in  $C^0(\Omega)$ , have been obtained by Campanato (see [3]). Without assuming continuity of coefficients, we mention the study by Acquistapace [1] where Campanato's results are refined considering that  $A_{ij}^{\alpha\beta}$  belongs to a class that neither contains nor is being contained by  $C^0(\Omega)$ , hence in general discontinuous. Moreover, we recall the study made by Huang in [14] where he shows the regularity of weak solutions of linear elliptic systems with coefficients  $A_{ij}^{\alpha\beta}(x) \in VMO$ . So, it seems to be natural to expect partial regularity results under the condition that the coefficients of the principal terms  $A_{ij}^{\alpha\beta} \in VMO$ , even for nonlinear cases. Daneček and Viszus in [6] consider the regularity of minimizer for the functional

$$\int_{\Omega} \{ A_{ij}^{\alpha\beta}(x) D_{\alpha} u^i D_{\beta} u^j + g(x, u, Du) \} dx$$

where the term  $g(x, u, Du)$  is such that

$$|g(x, u, z)| \leq f(x) + |z|^\gamma$$

where  $f \in L^p(\Omega)$ ,  $2 < p \leq \infty$ ,  $f \geq 0$  a. e. on  $\Omega$ ,  $L$  is a non-negative constant and  $0 \leq \gamma < 2$ .

They obtained hölder regularity of minimizer assuming that  $A_{ij}^{\alpha\beta}(x) \in VMO$ .

We also recall the paper [8] where the authors obtain regularity results for minimizers of the quasilinear functionals

$$\int_{\Omega} A_{ij}^{\alpha\beta}(x, u) D_{\alpha} u^i D_{\beta} u^j dx,$$

where the coefficients  $A_{ij}^{\alpha\beta}(x, u)$  have VMO dependence on the variable  $x$  and continuous dependence on  $u$ .

In the paper [21] we improve the last mentioned result in Morrey spaces because we have considered inside the integral the term  $g(x, u, Du)$  and the result by Danečěk and Viszus because we consider  $A_{ij}^{\alpha\beta}$  dependent not only on  $x$  but also on  $u$ . In the present note we extend our above cited regularity results because we consider the more general class Campanato spaces.

Before an outline of the proof we state a preliminary Lemma by Campanato.

**Lemma 2.6.** *Let  $B(x_0, R)$  be a fixed ball and  $u \in W^{1,2}(B(x_0, R); \mathbb{R}^N)$  be a weak solution of the system*

$$D_{\alpha}(A_{ij}^{\alpha\beta} D_{\beta} u^j) = 0, \quad i = 1, \dots, N$$

where  $A_{ij}^{\alpha\beta}$  are constant and satisfy the ellipticity condition. Then  $\forall t \in (0, 1]$

$$\int_{B(x_0, tR)} |Du|^2 dx \leq c \cdot t^n \int_{B(x_0, R)} |Du|^2 dx.$$

#### PROOF OF THE THEOREM

For simplicity we'll consider the case  $g = 0$ . Let  $R > 0$  and  $x_0 \in \Omega$  such that  $B(x_0, R) \subset \subset \Omega$ .

Let  $v$  be the minimum of the "freezing" functional  $\mathcal{A}$ , that is

$$\mathcal{A}_0(v, B(x_0, \frac{R}{2})) = \int_{B(x_0, \frac{R}{2})} A_{ij}^{\alpha\beta}(x_0, u_{\frac{R}{2}}) D_{\alpha} v^i D_{\beta} v^j dx$$



with  $v \equiv u$  on  $\partial B(x_0, \frac{R}{2})$ . The idea of freezing is the same used by Chiarenza, Frasca and Longo in [4].

For  $0 \leq \lambda < n$  and  $q \leq 2$  we have that

$$\begin{aligned} \|Du\|_{\mathcal{L}^{q,\lambda}(\Omega)} &= \|Du\|_{L^q(\Omega)} + [Du]_{q,\lambda} \leq \\ &\leq \mathcal{K} \|Du\|_{L^{q,\lambda}(\Omega)} \leq \mathcal{K} \|Du\|_{L^{2,\lambda}(\Omega)} \end{aligned}$$

where the constant  $\mathcal{K}$  is independent of  $u$ . We observe that  $A_{ij}^{\alpha\beta}(x_0, u_{\frac{R}{2}})$  are constant coefficients, then from the above Lemma,  $\forall t \in (0, 1]$ ,

$$\int_{B(x_0, t\frac{R}{2})} |Du|^2 dx \leq c \cdot t^n \int_{B(x_0, \frac{R}{2})} |Du|^2 dx.$$

Let  $w = u - v$ , then  $w \in W_0^{1,2}(B(x_0, \frac{R}{2}))$

$$\begin{aligned} &\int_{B(x_0, t\frac{R}{2})} |Du|^2 dx \leq \\ &\leq c \cdot \left\{ t^n \int_{B(x_0, \frac{R}{2})} |Du|^2 dx + \int_{B(x_0, \frac{R}{2})} |Dw|^2 dx \right\}. \end{aligned}$$

Let us estimate:

$$\int_{B(x_0, \frac{R}{2})} |Dw|^2 dx.$$

From the hypothesis and a Lemma in [12] we have

$$\nu \int_{B(x_0, \frac{R}{2})} |Dw|^2 dx \leq \left\{ \mathcal{A}^0(u, B(x_0, \frac{R}{2})) - \mathcal{A}^0(v, B(x_0, \frac{R}{2})) \right\}$$

adding and subtracting:

$$\begin{aligned} &A_{ij}^{\alpha\beta}(x, u_{\frac{R}{2}}) D_\alpha u^i D_\beta u^j, \quad A_{ij}^{\alpha\beta}(x, u_{\frac{R}{2}}) D_\alpha v^i D_\beta v^j \\ &A_{ij}^{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^j, \quad A_{ij}^{\alpha\beta}(x, v) D_\alpha v^i D_\beta v^j \end{aligned}$$

we obtain some different kinds of integrals that we now examine.

Using  $L^p$  estimate, we can estimate the terms with  $|Du|^2$  as follows:

$$\begin{aligned} \int_{B(x_0, \frac{R}{2})} |A_{ij}^{\alpha\beta}(x_0, u_{\frac{R}{2}}) - A_{ij}^{\alpha\beta}(x, u_{\frac{R}{2}})| \cdot |Du|^2 dx &\leq \\ &\leq c \{ \eta(A(\cdot, u_{\frac{R}{2}}); R) \}^{1-\frac{2}{p}} \int_{B(x_0, R)} |Du|^2 dx. \end{aligned}$$

We also observe that:

$$\begin{aligned} \int_{B(x_0, \frac{R}{2})} |A_{ij}^{\alpha\beta}(x, u_{\frac{R}{2}}) - A_{ij}^{\alpha\beta}(x, u)| D_\alpha u^i D_\beta u^j dx &\leq \\ &\leq c \left( \int_{B(x_0, R)} |Du|^2 dx \right) \left( \int_{B(x_0, \frac{R}{2})} \omega(|u_{\frac{R}{2}} - u|^2) dx \right)^{1-\frac{2}{p}} \leq \\ &\leq c \left( \int_{B(x_0, R)} |Du|^2 dx \right) \left( \omega \left( R^{2-n} \int_{B(x_0, \frac{R}{2})} |Du|^2 dx \right) \right)^{1-\frac{2}{p}}. \end{aligned}$$

Moreover, we can estimate the terms having  $|Dv|^2$  similarly using  $L^p$  estimates for  $Dv$ .

Then if  $\rho = tR$

$$\begin{aligned} \int_{B(x_0, \rho)} |Du|^2 dx &\leq \\ &\leq c \left\{ \left( \frac{\rho}{R} \right)^n + \left( \omega \left( R^{2-n} \int_{B(x_0, \frac{R}{2})} |Du|^2 dx \right) \right)^{1-\frac{2}{p}} \right. \\ &\quad \left. + \left( \eta(A(\cdot, u_{\frac{R}{2}}), R) \right)^{1-\frac{2}{p}} \right\} \cdot \left( \int_{B(x_0, \frac{R}{2})} |Du|^2 dx \right). \end{aligned}$$

Using a Lemma contained in [9] and selecting  $\rho$  sufficiently small, specifically  $\rho < \frac{R}{2}$ , we have

$$\int_{B(x_0, \rho)} |Du|^2 dx \leq c \cdot \rho^\lambda.$$

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